ELSEVIER

Discrete Mathematics 150 (1996) 453–456

DISCRETE MATHEMATICS

Note

On the number of irreducible coverings by edges of complete bipartite graphs

Ioan Tomescu

Faculty of Mathematics, University of Bucharest, Str. Academiei 14, 70109 Bucharest, Romania

Received 24 September 1993; revised 21 September 1994

Abstract

In this paper it is proved that the exponential generating function of the numbers, denoted by $N(p,q)$, of irreducible coverings by edges of the vertices of complete bipartite graphs $K_{p,q}$ equals $\exp(xe^y + ye^x - x - y - xy) - 1$.

Let $K_{p,q}$ be a complete bipartite graph. A set C of edges of $K_{p,q}$ is a covering of the vertices of $K_{p,q}$ if $\bigcup_{ab\in C}\{a,b\} = V(K_{p,q})$. This covering is said to be irreducible if, for any edge $ab \in C$, the set $C\backslash\{ab\}$ is not a covering of the graph, i.e., if any proper subset of C is not a covering. It is clear that a covering C of the vertex-set of bipartite graph is irreducible if and only if it does not contain three edges a_1b_1, a_2b_1, a_2b_2 generating a path a_1, b_1, a_2, b_2 of length three. Any irreducible covering of $K_{p,q}$ consists of some vertex-disjoint stars which contain at least two vertices each. The case when these stars may be reduced to a single vertex was considered by Farrell [1].

If we denote by $N(p,q)$ the number of irreducible coverings by edges of the vertices of $K_{p,q}$, we have [2]:

Lemma 1. *The numbers $N(p,q)$ satisfy the following recurrence formula:*

$$N(p,q) = \sum_{r=2}^{q-1}\binom{q}{r}N(p-1,q-r) + q\sum_{s=0}^{p-2}\binom{p-1}{s}N(p-s-1,q-1) \qquad (1)$$

and $N(1,p) = N(p,1) = 1$ for any $p \geqslant 1$.

Table 1 computes these numbers up to $p = q = 5$ [2].

SSDI 0012-365X(95)00215-4

Table 1

	q				
p	1	2	3	4	5
1	1	1	1	1	1
2	1	2	6	14	30
3	1	6	15	48	165
4	1	14	48	184	680
5	1	30	165	680	2945

Let

$$G(x,y)=\sum_{p,q=0}^{\infty}\frac{N(p,q)x^p y^q}{p!q!}=xy+\frac{xy^2}{2}+\frac{x^2y}{2}+\frac{x^2y^2}{2}$$
$$+\frac{xy^3}{6}+\frac{x^3y}{6}+\frac{xy^4}{24}+\frac{x^4y}{24}+\frac{x^2y^3}{2}+\frac{x^3y^2}{2}+\cdots$$

be the exponential generating function of the numbers $N(p,q)$.

Theorem 1. *The following equality holds:*

$$G(x,y)=\exp(xe^y+ye^x-x-y-xy)-1.$$

Proof. Let $F(x,y)=1+G(x,y)$. Because $N(p,0)=N(0,q)=0$ we can write (1) as follows:

$$N(p+1,q+1)=\sum_{r=2}^{q+1}\binom{q+1}{r}N(p,q+1-r)+(q+1)\sum_{s=0}^{p}\binom{p}{s}N(p-s,q). \quad (2)$$

By standard techniques (see [3]) we obtain that: the exponential generating function of the numbers $N(p+1,q+1)$ is $\partial^2F/\partial x\,\partial y$;

$$\sum_{r=2}^{q+1}\binom{q+1}{r}N(p,q+1-r)$$

is $e^y(\partial F/\partial y)+e^yF-\partial F/\partial y-F-y(\partial F/\partial y)$ since

$$\sum_{r=2}^{q+1}\binom{q+1}{r}N(p,q+1-r)$$
$$=\sum_{r=0}^{q+1}\binom{q+1}{r}N(p,q+1-r)-N(p,q+1)-(q+1)N(p,q)$$

and

$$\binom{q+1}{r} = \binom{q}{r} + \binom{q}{r-1};$$

$$q\sum_{s=0}^{p}\binom{p}{s}N(p-s,q)$$

is $ye^x(\partial F/\partial y)$;

$$\sum_{s=0}^{p}\binom{p}{s}N(p-s,q)$$

is e^xF, respectively.

Therefore (2) is translated into the partial differential equation:

$$\frac{\partial^2 F}{\partial x\,\partial y} = (e^x + e^y - 1)F + (ye^x + e^y - y - 1)\frac{\partial F}{\partial y}. \tag{3}$$

By the substitution $F = e^f$ we get

$$\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial x\,\partial y} = e^x + e^y - 1 + (ye^x + e^y - y - 1)\frac{\partial f}{\partial y}. \tag{4}$$

If $\partial f/\partial x = ye^x + e^y - y - 1$ then $\partial^2 f/\partial x\,\partial y = e^x + e^y - 1$ and (4) is verified. It follows that $f = xe^y + ye^x - xy - x + C(y)$. Since $F(0,0) = 1$ and $\partial F/\partial y|_{x=0} = 0$ we obtain that $C(0) = 0$ and $C'(y) = -1$, hence $C(y) = -y$, which implies that $F(x,y) = \exp(xe^y + ye^x - xy - x - y)$. □

Corollary 1. *For every $p,q \geqslant 1$ the following equality holds:*

$$N(p,q) = p!q!\sum_{k=1}^{\min(p,q)}\sum_{(\alpha_i)(\beta_j)}\frac{1}{\alpha_1!\cdots\alpha_p!\beta_1!\cdots\beta_q!(k-\sum\alpha_i-\sum\beta_j)!}$$
$$\times\prod_{i=1}^{p}\frac{1}{((i+1)!)^{\alpha_i}}\prod_{j=1}^{q}\frac{1}{((j+1)!)^{\beta_j}}, \tag{4}$$

where the second sum is over all solutions of $\sum_{i=1}^{p} i\alpha_i = p-k$; $\sum_{j=1}^{q} j\beta_j = q-k$; $\alpha_i, \beta_j \geqslant 0$ for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ and $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j \leqslant k$.

Proof. From Theorem 1 it follows that $N(p,q)/p!q!$ is the coefficient of x^py^q in the development

$$\exp\left(xy + x\sum_{i=2}^{q}\frac{y^i}{i!} + y\sum_{j=2}^{p}\frac{x^j}{j!}\right) - 1$$
$$= \sum_{k=1}^{\infty}\frac{1}{k!}\left(xy + x\sum_{i=2}^{q}\frac{y^i}{i!} + y\sum_{j=2}^{p}\frac{x^j}{j!}\right)^k$$
$$= \sum_{k=1}^{\infty}\frac{x^ky^k}{k!}\left(1 + \sum_{i=1}^{q}\frac{y^i}{(i+1)!} + \sum_{j=1}^{p}\frac{x^j}{(j+1)!}\right)^k.$$

By the multinomial formula

$$\left(1+\sum_{i=1}^{q}\frac{y^i}{(i+1)!}+\sum_{j=1}^{p}\frac{x^j}{(j+1)!}\right)^k$$
$$=\sum(\gamma,\alpha_1,\ldots,\alpha_p^k,\beta_1,\ldots,\beta_q)\,x^{\alpha_1+2\alpha_2+\cdots+p\alpha_p}y^{\beta_1+2\beta_2+\cdots+q\beta_q}$$
$$\times\left(\frac{1}{2!}\right)^{\alpha_1}\cdots\left(\frac{1}{(p+1)!}\right)^{\alpha_p}\left(\frac{1}{2!}\right)^{\beta_1}\cdots\left(\frac{1}{(q+1)!}\right)^{\beta_q},$$

where the sum is over $\gamma+\alpha_1+\cdots+\alpha_p+\beta_1+\cdots+\beta_q=k$ and $\gamma=k-\sum_{i=1}^{p}\alpha_i-\sum_{j=1}^{q}\beta_j\geqslant;\ \alpha_1,\ldots,\alpha_p,\ \beta_1,\ldots,\beta_q\geqslant 0$ and (4) follows. □

References

[1] E.J. Farrell, On a class of polynomials associated with the stars of a graph and its application to node-disjoint decompositions of complete graphs and complete bipartite graphs into stars, Canad. Math. Bull. 22 (1979) 35–46.

[2] I. Tomescu, Some properties of irreducible coverings by cliques of complete multipartite graphs. J. Combin. Theory Ser. B 28 (1980) 127–141.

[3] H. Wilf, Generating functionology (Academic Press, New York, 1990).